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TOWARD A THEORY OF THE PROPAGATION OF ELASTO-PLASTIC WAVES IN STRAIN-HARDENING MEDIA

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A model of elastoplastic flow with isotropic and translational strain-hardening expressed as a variational inequality is constructed to obtain an integral generalization that makes it possible to study the class of discontinuous solutions. Generalized solutions corresponding to different types of strain-hardening are compared for the problem of the propagation of plane shear waves.

Mandel [1] was the first to examine the construction of generalized solutions in dynamic problems of the Prandtl–Reuss theory of elastoplastic flow, but he mistakenly concluded that velocity and stress discontinuity fronts cannot be described unambiguously in this theory. A complete system of relations for strong discontinuities was obtained in [2] on the basis of considerations pertaining to maximum plastic energy dissipation on a front in a model of linear isotropic and translational strain-hardening.

It was shown in [3] that the system of quasilinear Prandtl–Reuss equations corresponding to the elastic–ideally-plastic model cannot be reduced to divergent form. It is thus impossible to generalize it in the form of a complete system of integral conservation laws and construct discontinuous solutions — as is possible for models of ideal media [4].

In the present study, we propose an integral formulation that is equivalent to the initial equations of flow theory for an arbitrary strain-hardening curve. We then make use of this formulation to write relations for strongly discontinuous solutions, without resort to any additional considerations.

1. Variational Inequality for a Model of a Strain-Hardening Body. Within the framework of a geometrically linear approximation, the model of an elastoplastic body can be represented in the form of a system of equations of motion, Hooke's law, and the principle of the maximum of the rate of energy dissipation:

$$\rho v_{i,t} = \sigma_{ij,j},\ e^0_{ij} = a_{ijkl}\sigma_{kl,t},\ (\bar{\sigma}_{ij} - \sigma_{ij})e^p_{ij} \leq 0,\ \frac{1}{2}(v_{i,j} + v_{j,i}) = e^0_{ij} + e^p_{ij}. \tag{1.1}$$

Here, ρ is density; v_i is the velocity vector relative to a cartesian coordinate system; a_{ijkl} is the tensor of the elastic compliance moduli, which is symmetric and positive-definite; e^0_{ij} and e^p_{ij} are the elastic and plastic components of the strain-rate tensor. The principle of the maximum is satisfied for an arbitrary variation of the stress tensor subject to the limitation

$$f(\sigma_{ij} - \tau_{ij}) \leq \theta, \tag{1.2}$$

where $f = f(\sigma_{ij})$ is the convex positive-uniform flow function of the undeformed material; τ_{ij} is the symmetric microstress tensor; θ is the variable yield point.

Inequality (1.2) describes classical strain-hardening variants: isotropic strain-hardening corresponds to $\tau_{ij} = 0$, while translational strain-hardening corresponds to $\theta = \theta_s = \text{const}$. For the model of a elastic–ideally-plastic medium, $\tau_{ij} = 0$, $\theta = \theta_s$. In the general case, system (1.1) must be supplemented by equations describing the evolution of parameters τ_{ij} and θ:

$$\xi_{ij,t} = e^p_{ij},\ \theta\eta_{,t} = (\sigma_{ij} - \tau_{ij})e^p_{ij}. \tag{1.3}$$

The plastic strain tensor ξ_{ij} and the scalar coefficient η entering into these equations are assigned functions of the strain-hardening parameters. Meanwhile,

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$$\xi_{ij} = \partial\Phi^p/\partial\tau_{ij},\ \eta = \partial\Phi^p/\partial\theta$$

($\Phi^p = \Phi^p(\tau_y, \theta)$ is the plastic Gibbs potential).

System (1.1), (1.3) is equivalent to the inequality

$$\begin{gathered}(v_i^* - v_i)(\rho v_{i,t} - \sigma_{ij,j}) + (\sigma_{ij}^* - \sigma_{ij}) \\ \times (a_{ijkl}\sigma_{kl,t} - v_{i,j}) + (\tau_{ij}^* - \tau_{ij})\xi_{ij,t} + (\theta^* - \theta)\eta_{,t} \geqslant 0,\end{gathered} \tag{1.4}$$

in which the variation of the velocity vector is arbitrary, while the variations of the stress tensor and the strain-hardening parameters satisfy limitation (1.2). In fact, all of the relations (1.1), (1.3) can be obtained by specially choosing the allowable variations in (1.4). In particular, (1.1) follows when $\sigma_{ij}^* - \bar{\sigma}_{ij}$, $\tau_{ij}^* = \tau_{ij}$, $\theta^* = \theta$. The system of equations describing the evolution of τ_{ij} follows when $\sigma_{ij}^* = \sigma_{ij} + \gamma_{ij}$, $\tau_{ij}^* = \tau_{ij} + \gamma_{ij}$, where γ_{ij} is an arbitrary symmetric tensor. The last equation of (1.3) is obtained when $\sigma_{ij}^* = \tau_{ij}^* = \theta^* = 0$ and $\sigma_{ij}^* = 2\sigma_{ij}$, $\tau_{ij}^* = 2\tau_{ij}$, $\theta^* = 2\theta$. On the other hand, by assuming in (1.1) that

$$\bar{\sigma}_{ij} = (\sigma_{ij}^* - \tau_{ij}^*)\frac{\theta}{\theta^*} + \tau_{ij},$$

and allowing for (1.3), we derive (1.4).

Inequality (1.4), representing one of the equivalent formulations of flow theory, is a special case of a hyperbolic variational inequality with a nonlinear operator:

$$\begin{gathered}(\mathrm{u}^* - \mathrm{u})(\mathrm{N}\langle \mathrm{u} \rangle - \mathrm{g}) \geqslant 0 \quad \mathrm{u}, \mathrm{u}^* \in \mathrm{K}, \\ \mathrm{N}\langle \mathrm{u} \rangle = \partial\varphi/\partial t - \sum_{s=1}^{n} \partial\psi_s/\partial x_s.\end{gathered} \tag{1.5}$$

Here, $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ is an unknown m-dimensional vector function; $\mathbf{K} = \mathbf{K}(t, \mathbf{x})$ is the convex set of allowable variations of the solution; $\varphi = \partial\Phi/\partial u$, $\psi_s = \partial\Psi_s/\partial u$, $\Phi = \Phi(t, \mathbf{x}, \mathbf{u})$ and $\Psi_s = \Psi_s(t, \mathbf{x}, \mathbf{u})$ are assigned scalar potentials; $\mathbf{g} = \mathbf{g}(t, \mathbf{x}, \mathbf{u})$ is an assigned vector function. The symbols $\partial/\partial t$ and $\partial/\partial x_s$ denote total derivatives with respect to the corresponding variables. We assume below that φ and ψ_s are continuously differentiable and that $\mathbf{K}$ and $\mathbf{g}$ are continuous functions of their arguments.

In the model being considered, $\mathbf{u}$ contains components of the velocity vector, stress tensor, and strain-hardening parameters

$$\Phi = \Phi^p + \frac{1}{2}\rho v_i v_i + \frac{1}{2} a_{ijkl}\sigma_{ij}\sigma_{kl},\ \Psi_s = \sigma_{is}v_i,$$

The vector $\mathbf{g}$ is equal to zero, while the set $\mathbf{K}$ is determined by limitation (1.2).

In the case when the function $\mathbf{\Phi}$ is strictly convex and $\mathbf{g}$ satisfies the Lipshitz condition with respect to the variable $\mathbf{u}$, arbitrary estimates that generalize estimates of the solutions of quasilinear hyperbolic systems of equations in characteristic conoids are valid for inequality (1.5). These estimates prove that the "short-term" solutions of the Cauchy problem

$$\mathrm{u}|_{t=0} = \mathrm{u}_0(\mathrm{x})$$

are unique and independent of the initial data, as are the solutions of mixed problems with dissipative boundary conditions. The estimates also prove that the region in which the solutions are obtained is finite (i.e. they prove that the rate of propagation of perturbations in the corresponding models is finite — see [5]).

2. Relations for a Strongly Discontinuous Solution. In the general case, a variational equality can be written in divergent form

$$u^*N\langle u\rangle - (u^* - u)g \geqslant \partial(u\varphi - \Phi)/\partial t$$
$$- \sum_{s=1}^{n} \partial(u\psi_s - \Psi_s)/\partial x_s + h,\ h(t, x, u) = \Phi_{,t} - \sum_{s=1}^{n} \Psi_{s,s}$$

and thus has an integral generalization that is equivalent to it for smooth solutions:

$$\begin{aligned}&\iint_G \{-\varphi(\chi u^*)_{,t} + \sum_{s=1}^{n}\psi_s(\chi u^*)_{,s} - (u^* - u)\chi g\} d\omega_x dt \\ &\geqslant \iint_G \{-(u\varphi - \Phi)\chi_{,t} + (u\psi_s - \Psi_s)\chi_{,s} + h\} d\omega_x dt.\end{aligned} \tag{2.1}$$

Here, $\chi \in C^\infty(G)$ is an arbitrary non-negative function that is finite in the region G; $\mathbf{u}^* = \mathbf{u}^*(t, x) \in \mathbf{K}$ is a continuously differentiable vector function.

The integral inequality naturally determines the class of generalized solutions that contains all of the vector functions $\mathbf{u} \in \mathbf{L}_1(G)$ satisfying (2.1) for all allowable $\mathbf{u}^*$. In particular, this class contains solutions with a strong discontinuity, as well as solutions that have a first-order discontinuity on a certain hypersurface S and are continuously differentiable in the rest of G.

By applying Green's formula to integrals taken over the continuous subregions of such a solution and by allowing for the arbitrariness of χ, we can show that the following inequality is valid at points of the hypersurface S

$$u^*[r] \geqslant c[u\varphi - \Phi] + \sum_{s=1}^{n} [u\psi_s - \Psi_s]\nu_s,\ r = c\varphi + \sum_{s=1}^{n} \psi_s\nu_s,$$

where $c \geq 0$ is the velocity of the front of the discontinuity — which is the section S of the hyperplane $t = \text{const}$ — in the direction of its normal ν_s; the brackets denote the discontinuity of the function.

The last inequality is rewritten in the equivalent form

$$(u^* - u^0)[r] \geqslant d \equiv c(\varphi^0[u] - [\Phi]) + \sum_{s=1}^{n} (\psi_s^0[u] - [\Psi_s])\nu_s. \tag{2.2}$$

Here, $\mathbf{u}^0 = (\mathbf{u}^+ + \mathbf{u}^-)/2$ ($\mathbf{u}^\pm$ are unilateral limits of the solution on S). The quantities φ^0 and ψ^0_s are similarly determined. Assuming in (2) that $\mathbf{u}^* = \mathbf{u}^0 \in \mathbf{K}$, we can obtain the condition for realization of the discontinuity ($d \leq 0$). This condition plays the same role for inequality (1.5) as does the condition of non-negativity of the entropy discontinuity in models of ideal media.

By expanding the discontinuities $[\boldsymbol{\Phi}]$ and $[\boldsymbol{\Psi}]_s$ into Taylor series, we can easily show that $d/[\mathbf{u}]^2 \to 0$ as $[\mathbf{u}] \to 0$, i.e. d is of third-order smallness compared to $[\mathbf{u}]$.

We will henceforth focus on two types of solutions with a strong discontinuity: regular solutions, for which the left side of inequality (2.2) is non-negative at points of S for all $\mathbf{u}^* \in \mathbf{K}$; irregular solutions, for which there exists a vector $\mathbf{u}'$ $\in \mathbf{K}$ such that $(\mathbf{u}' - \mathbf{u}' - \mathbf{u}^0)[\mathbf{r}] < 0$ at a certain point S.

We have the following theorem: on the front of discontinuity of the regular solution for all $\lambda(\,|\,\lambda\,|\, \leq 1/2$, in this case $\mathbf{u}^\lambda = (\lambda + 1/2)\mathbf{u}^+ + (1/2 - \lambda)\mathbf{u}^- \in \mathbf{K})$

$$(u^* - u^\lambda)[r] \geqslant 0. \tag{2.3}$$

In fact, by virtue of the definition of a regular solution, inequality (2.3) is satisfied when $\lambda = 0$. First taking the vector $\mathbf{u}^+$ and then $\mathbf{u}^-$ as $\mathbf{u}^*$ in (2.3) and summing the resulting inequalities, we can establish that $[\mathbf{u}][\mathbf{r}] = 0$. It follows from this, with allowance for the formula $\mathbf{u}^\lambda = \mathbf{u}^0 + \lambda[\mathbf{u}]$, that inequality (2.3) is valid for any λ.

From a geometric viewpoint, the above theorem means that if any point $\mathbf{u}^\lambda$ of a segment in an m-dimensional state with the ends $\mathbf{u}^+$, $\mathbf{u}^-$ lies completely within the set $\mathbf{K}$, then $[\mathbf{r}] = 0$ by virtue of the arbitrariness of the variation $\mathbf{u}^* - \mathbf{u}^\lambda$ in (2.3). If the entire segment is on the boundary of $\mathbf{K}$, then the vector $[\mathbf{r}]$ will be directed along its inner normal for all points of the segment.

Thus, regular solutions in the theory of elastoplastic flow can contain two types of discontinuities: elastic waves, determined by the system of equations [**r**] = 0; plastic waves, corresponding to the condition of orthogonality with respect to the yield surface. It should be noted that the functions Φ and Ψ_s will be quadratic (d = 0) in the case of an elastoplastic medium, so that any discontinuous solution will be regular. The complete system of strong discontinuity relations that follows from (2.3) for the elastic-ideally-plastic model were obtained in [6, 7] by the same method examined here.

3. Propagation of Shear Stress Waves. Generalized solutions with a strong discontinuity can be constructed in closed form in the unidimensional problem of propagation of plane shear waves generated by the application of a shear stress $\sigma_{13} = q$ to the surface of a half-space $x_1 \geq 0$. In this problem, variational inequality (2.3) and the limitation on the variation take the form

$$(\sigma_{13}^* - \sigma_{13}^\lambda)\left(\frac{1}{G} - \frac{1}{\rho c^2}\right)[\sigma_{13}] + 2(\tau_{13}^* - \tau_{13}^\lambda)[\xi_{13}] + (\theta^* - \theta^\lambda)[\eta] \geqslant 0,\ \sigma_{13}^* - \tau_{13}^* \leqslant \theta^* \tag{3.1}$$

(G is the shear modulus).

After analyzing the system of equations obtained when the rule for Lagrangian multipliers (the Kuhn—Tucker theorem) is applied to inequality (1.4)

$$\rho v_{3,t} = \sigma_{13,1},\ \frac{1}{G}\sigma_{13,t} - v_{3,1} + \eta_{,t} = 0,\ 2\xi_{13,t} = \eta_{,t} \geqslant 0, \tag{3.2}$$

we find that the strain-hardening parameters τ_{13} and θ are dependent on one another in the case of pure shear. The are connected by the equality $\eta = 2\xi_{13}$. This equation, together with the stress-strain curve of the material $(2\xi_{13} + \sigma_{13}/G = F(\sigma_{13})$, makes it possible to determine the plastic potential

$$\Phi^p = \int 2\xi_{13} d\tau_{13} + \eta d\theta = \int_0^{\tau_{13}+\theta} \{F(\sigma) - \sigma/G\} d\sigma.$$

With a constant value $q > 0$, the regular generalized solution of the problem contains two lines of discontinuity in the plane of the variables t, x_1: the elastic precursor $x_1 = c_s t$, propagating with the velocity of the transverse waves $c_s = \sqrt{G/\rho}$; the plastic wave $x_1 = c_\tau t$. In the region between the discontinuities, $\sigma_{13} = \theta_s$. Behind the front of the plastic wave, $\sigma_{13} = q$. With allowance for the Lagrangian multiplier rule, the velocity of the plastic wave can be found from variational inequality (3.1) in the form

$$c_\tau = \left(\frac{1}{\rho}\frac{q - \theta_s}{F(q) - F(\theta_s)}\right)^{1/2}.$$

The condition under which the plastic discontinuity can be realized leads to the inequality

$$\int_{\theta_s}^{q} \sigma dF(\sigma) \leqslant \{F(q) - F(\theta_s)\}(q + \theta_s)/2,$$

which has a simple geometric interpretation and is satisfied automatically for stress–strain curves that are convex-downward. In the case of curves that are convex-upward, system (3.2) has a self-similar solution that depends on the variable $c = x_1/t$ and has one line of discontinuity: $x_1 = c_s t$. In the neighborhood of a centered wave, this solution is determined by the usual equation (see [8], for example)

$$\rho c^2 dF/d\sigma_{13} = 1.$$

The curve that describes pure shear for a wide range of elastoplastic materials has a convex-downward section corresponding to the yield point plateau and a convex upward section corresponding to active strain-hardening. The condition under which such a curve is obtained prohibits the formation of two plastic discontinuities propagating at the same or different (Fig. 1) velocities. Given a sufficiently large value of the external stress q, the line of plastic discontinuity corresponds to the point θ_c at which a ray originating from the point corresponding to the limiting equilibrium state touches the stress–strain curve.

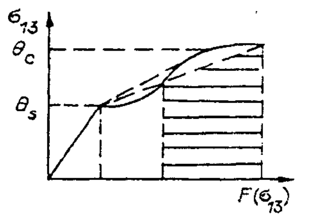

Fig. 1

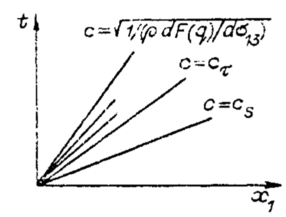

Fig. 2

The value of c_τ is calculated from the same formula after q is replaced by θ_c. At $q > \theta_c$, the plastic discontinuity adjoins a self-similar solution dependent on $c = x_1/t$ (Fig. 2).

Thus, the regular discontinuous solution describing the propagation of shear loading waves is determined completely by the form of the function F and is independent of the type of strain-hardening.

4. Steady Stability of Shear Waves. Together with the regular solutions to the problem of the propagation of shear stress waves, there may also be a set of irregular discontinuous solutions. These solutions can be obtained after replacing the equality $\sigma_{13} - \tau_{13} = \theta$ behind the front of the plastic discontinuity by a strict inequality. Such a substitution violates variational inequality (3.1) for certain variations of the shear stress and the strain-hardening parameters. The problem of choosing a natural discontinuous solution can be solved by the viscosity method [9].

Let us examine the system of equations describing the propagation of viscoelastic-viscoplastic shear waves in a strain-hardening medium:

$$\rho v_{3,t} = \sigma_{13,1} + \mu_0 v_{3,11}, \quad \frac{1}{G}\sigma_{13,t} - v_{3,1} + \eta_{,t} = 0,$$
$$2\xi_{13,t} = \eta_{,t} = \max\{0, \sigma_{13} - \tau_{13} - \theta\}/\mu_p. \tag{4.1}$$

Here, μ_0 and μ_p are the Kelvin–Voigt and Shvedov–Bingham viscosity coefficients. System (4.1) changes into (3.2) as μ_0 and μ_p approach zero, so it is natural to require that the generalized solutions of the elastoplastic model be stable in the sense of allowing passage to the limit for the viscosity parameters in the sequence of solutions of system (4.1).

Since the strain-hardening parameters are again dependent variables for a viscoelastic–viscoplastic model, then $\tau_{13} + \theta = H(\eta)$. Here, the function $H(\eta)$ can be found from the curve for pure shear as the solution of the equation $\eta + H/G = F(H)$. Differentiation of this equation twice leads to the equation

$$\left(\frac{1}{G} - \frac{dF}{d\sigma_{13}}\right)\frac{d^2H}{d\eta^2} = \left(\frac{dH}{d\eta}\right)^2 \frac{d^2F}{d\sigma_{13}^2},$$

With allowance for the inequality $1/G < dF/d\sigma_{13}$, it follows from this equation that the convexity of $H(\eta)$ is unambiguously connected with the convexity of the strain-hardening diagram.

In the steady-state case, system (4.1) leads to two ordinary differential equations of the form

$$\mu_0 c\, d\sigma_{13}/dy = P(\sigma_{13}, \eta), \quad \mu_0 c\, d\eta/dy = Q(\sigma_{13}, \eta), \tag{4.2}$$

where $y = ct - x_1$; $Q(\sigma_{13}, \eta) = \mu_0 \max\{0, \sigma_{13} - H(\eta)\}/\mu_p$;

$$P(\sigma_{13}, \eta)/G = -(1 - \rho c^2/G)(\sigma_{13} - C) + \rho c^2 \eta - Q(\sigma_{13}, \eta)$$

(C is an arbitrary constant of integration). It can be proven that if the functions $\sigma_{13}(y)$ and $\eta(y)$ form the solution of system (4.2), then the functions $\sigma_{13}(y/\varepsilon)$ and $\eta(y/\varepsilon)$ are the solution of the analogous system of equations with the viscosity parameters $\mu_0\varepsilon$ and $\mu_p\varepsilon$. Thus, passing to the limit for $\mu_0 \to 0$ (μ_p/μ_0 = const) is equivalent to $\varepsilon \to 0$ or $|y| \to \infty$, and the study of solutions that are steady and stable in terms of viscosity reduces to the analysis of the integral curves of system (4.2) corresponding to solutions with an infinite domain $-\infty < y < \infty$.

In accordance with the qualitative theory of differential equations [10], any unclosed integral curve of system (4.2) corresponding to a solution with an infinite range joins the singular points of the system

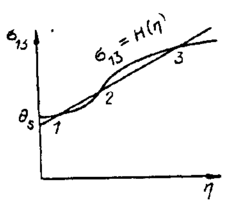

Fig. 3

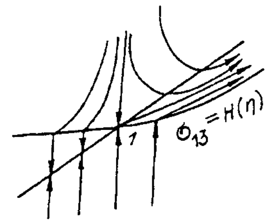

Fig. 4

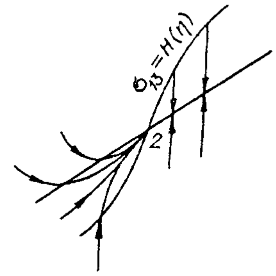

Fig. 5

$$P(\sigma_{13}^{\pm}, \eta^{\pm}) = Q(\sigma_{13}^{\pm}, \eta^{\pm}) = 0.$$

In the plane of the variables σ_{13}, η, these points correspond to points on the straight line $\sigma_{13} = \varkappa\eta + C$, $\varkappa = \rho c^2 G/(G - \rho c^2)$, located below the graph of the function $\sigma_{13} = H(\eta)$ (Fig. 3).

The given system is linear at $\sigma_{13} < H$. Its integral curves are segments of straight lines parallel to the axis σ_{13}. At $\sigma_{13} \geq H$, the matrix of the Jacobian of the system is equal to

$$\frac{\mu_0}{\mu_p}\begin{pmatrix} \mu_p(\rho c^2 - G)/\mu_0 - G & G(\mu_p \rho c^2/\mu_0 + dH/d\eta) \\ 1 & - dH/d\eta \end{pmatrix}.$$

The eigenvalues of this matrix

$$2\lambda_{1,2} = -\left\{G - \rho c^2 + \frac{\mu_0}{\mu_p}\left(G + \frac{dH}{d\eta}\right)\right\} \pm \sqrt{\left\{G - \rho c^2 - \frac{\mu_0}{\mu_p}\left(G + \frac{dH}{d\eta}\right)\right\}^2 + 4G^2\frac{\mu_0}{\mu_p}}$$

are always real-valued. The end points of the singular lines of system (4.2) are simple singular points when the condition of non-degeneracy $dH/d\eta \neq \varkappa$ is satisfied. If $dH/d\eta < \varkappa$, then $\lambda_1\lambda_2 < 0$, i.e. the corresponding point is a saddle point (points 1 and 3 in Fig. 3). If $dH/d\eta > \varkappa$ ($\lambda_1\lambda_2 > 0$), then the point is a stable node (point 2).

An analysis of the direction field makes it possible to construct an approximate representation of the integral curves in the neighborhood of each type of singular point (Figs. 4 and 5).
Using a standard approach [9, pp. 238-239], we can prove that there is a single integral curve that originates from point 1 as the separatrix of a saddle point and reaches node 2. It can also be proven that there are no integral curves corresponding to any other singular points. Thus, the unique steady-state solution obtained with fixed c and C corresponds to state 1 "ahead of the front" and state 2 "behind the front" of the plastic discontinuity. This solution is realized only when the curve for pure shear has a convex downward section and is characterized by the relations

$$[\sigma_{13}] = \varkappa[\eta], \ \sigma_{13}^{\pm} = H(\eta^{\pm}).$$

The relations obtained here show that the class of stable, steady solutions coincides with the class of regular plastic waves of strong discontinuity. Thus, irregular solutions describing the propagation of shear stress waves in an elastoplastic half-space do not satisfy the criterion of stationariness of plastic discontinuities.

REFERENCES

1. J. Mandel, "Plastic waves in an infinite three-dimensional medium," in: Mechanics [Russian translation], No. 5, Moscow (1963).
2. G. I. Bykovtsev and L. D. Kretova, "Propagation of shock waves in elastoplastic media," Prikl. Mat. Mekh., **36**, No. 2 (1972).
3. V.N. Kukudzhanov, "Study of equations of the dynamics of elastoplastic bodies with finite strains," in: Nonlinear Strain Waves, Tallinn (1978).
4. L. I. Sedov, Continuum Mechanics, Vol. 1, Nauka, Moscow (1976).

5. V.M. Sadovskii, "Hyperbolic variational inequalities in problems of the dynamics of elastoplastic media," Prikl. Mat. Mekh., **55**, No. 6 (1991).
6. V. M. Sadovskii, "Dynamic correctness of the theory of elastic-ideally-plastic flow," Din. Sploshnoi Sredy, No. 63 (1983).
7. W. J. Drugan and Yinong Shen, "Restrictions on dynamically propagating surfaces of strong discontinuity in elastic-plastic solids," J. Mech. Phys. Solids, **35**, No. 6 (1987).
8. V. K. Novatsky, Wave Problems in the Theory of Plasticity [Russian translation], Mir, Moscow (1978).
9. B. L. Rozhdestvenskii and N. N. Yanenko, Systems of Quasilinear Equations [in Russian], Nauka, Moscow (1978).
10. N. N. Bautin and E. A. Leontovich, Methods and Techniques for the Qualitative Study of Dynamic Two-Dimensional Systems [in Russian], Nauka, Moscow (1978).